TD 4. Matrices de densité, no-signalling

1 Trace

Definition 1 (Trace).
The trace of a matrix $A \in M_d(\mathbb{C})$ *is defined to be :* $Tr(A) = \sum_{i\in 0\ldots d-1} A_{ii}$. *Alternatively if* $\{|i\rangle\}$ *is the canonical o.n.b. of* $\mathbb{C}^d$ *:* $Tr(A) = \sum_{i\in 0\ldots d-1}\langle i|A|i\rangle$.

Lemma 1 (Cyclicity, linearity).
Let $A, B \in M_d(\mathbb{C})$. *We have*
$Tr(AB) = Tr(BA)$ *and* $Tr(A + B) = Tr(A) + Tr(B)$. *Similarily* $Tr(zA) = z\,Tr(A)$.

Proof.

$$\begin{aligned}
\mathrm{Tr}(AB) &= \sum_{ij} A_{ij}B_{ji} \\
&= \sum_{ij} B_{ji}A_{ij} = \sum_{ij} B_{ij}A_{ji} \\
&= \mathrm{Tr}(BA) \\
\mathrm{Tr}(A+B) &= \sum_i (A_{ii} + B_{ii}) = (\sum_i A_{ii}) + (\sum_i B_{ii}) \\
&= \mathrm{Tr}(A) + \mathrm{Tr}(B) \\
\mathrm{Tr}(zA) &= \sum_i zA_{ii} = z\sum_i A_{ii} = z\mathrm{Tr}(A) \qquad \square
\end{aligned}$$

Lemma 2.
Let $A \in M_d(\mathbb{C})$. *We have :*
$Tr(A|\psi\rangle\langle\psi|) = \langle\psi|A|\psi\rangle$.

Proof. Let $\{|i\rangle\}$ be the canonical o.n.b. Then :

$$\begin{aligned}
\mathrm{Tr}(A|\psi\rangle\langle\psi|) &= \sum_i \langle i|A|\psi\rangle\langle\psi|i\rangle \\
&= \sum_i \langle\psi|i\rangle\langle i|A|\psi\rangle = \langle\psi|\sum_i |i\rangle\langle i|A|\psi\rangle \\
&= \langle\psi|\mathbb{I}A|\psi\rangle = \langle\psi|A|\psi\rangle \qquad \square
\end{aligned}$$

2 Postulats revisités

Postulate 3 (Generalized measurements).
A generalized measurement upon an n-dimensional quantum system is described by a collection $\{M_k\}$ *of measurement operators* $M_k \in M_{m\times n}(\mathbb{C})$ *satisfying the completeness relation*

$$\sum_k M_k^\dagger M_k = \mathbb{I}.$$

If the quantum system has state $|\psi\rangle$, then the probability that result k occurs is given by

$$p(k) = Tr(M_k^\dagger M_k |\psi\rangle\langle\psi|) = \langle\psi|M_k^\dagger M_k|\psi\rangle.$$

Then state of the system after the measurement is

$$\begin{aligned}|\psi'\rangle &= M_k|\psi\rangle/\sqrt{\langle\psi|M_k^\dagger M_k|\psi\rangle}\\ &= M_k|\psi\rangle/\sqrt{p(k)}\end{aligned}$$

Remark 1. *This postulate has unitary evolutions as a subcase. This is the case when the collection has only one measurement operator.*

Remark 2. *This postulate has projective measurements as a subcase. This is the case when :*
$\forall kl \; M_k M_l = \delta_{kl} M_k$.

3 Distributions de probabilités d'états quantiques

We may want to mix 'classical' probabilities with 'quantum' amplitudes. What may seem a legitimate way to do this is to denote by $\{(p_0, |\psi_0\rangle), \ldots, (p_r, |\psi_r\rangle)\}$ the state of a quantum system which has state $|\psi_0\rangle$ with probability p_0, state $|\psi_1\rangle$ with probability p_1,...

Definition 2 (Ensemble state)**.** *An ensemble state is a list* $\{(p_0, |\psi_0\rangle), \ldots, (p_r, |\psi_r\rangle)\}$ *where*
$\forall i \; |\psi_i\rangle$ *is a normalized vector of* $\mathbb{C}^n$
and $\sum_i p_i = 1$.

The main weakness of ensemble states are their non-canonicity. That is we may have two different ensemble states which are physically undistinguishable, and hence 'equal for all purpose'.

Example 1. *The ensembles* $\{(1/2, |0\rangle), (1/2, |1\rangle)\}$ *and* $\{(1/2, 1/\sqrt{2}(|0\rangle + |1\rangle), (1/2, 1/\sqrt{2}(|0\rangle - |1\rangle))\}$ *are undistinguishable physically.*

Example 2. *The ensembles* $\{(3/4, |0\rangle), (1/4, |1\rangle)\}$ *and* $\{(1/2, \sqrt{3/4}|0\rangle + \sqrt{1/4}|1\rangle), (1/2, \sqrt{3/4}|0\rangle - \sqrt{1/4}|1\rangle)\}$ *are undistinguishable physically.*

To prove these things we need the theoretical tools provided next. Meanwhile I challenge you to find a generalized measurement $\{M_k\}$ which discriminates these ensembles (i.e. such that the $p(k)$ vary from one ensemble to the other). This is impossible.

4 Formalisme des matrices de densité

Postulate 1. *'*
The state of an n-dimensional quantum system is fully described by its density matrix ρ, which its a positive unit trace matrice over $\mathbb{C}^n$.
In other words $\rho \in Herm_d^+(\mathbb{C})$ and $Tr(\rho) = 1$.

Postulate 3. *'*
A generalized measurement upon an n-dimensional quantum system is described by a collection $\{M_k\}$ of measurement operators $M_k \in M_{m\times n}(\mathbb{C})$ satisfying the completeness relation

$$\sum_k M_k^\dagger M_k = \mathbb{I}.$$

If the quantum system has density matrix ρ, then the probability that result k occurs is given by

$$p(k) = Tr(M_k^\dagger M_k \rho).$$

Then state of the system after the measurement is

$$\begin{aligned} \rho' &= M_k \rho M_k^\dagger / Tr(M_k^\dagger M_k \rho) \\ &= M_k \rho M_k^\dagger / p(k) \end{aligned}$$

Postulate 4. *'*
The state space of a composite physical system is the tensor product of the space space of the component physical systems.
Before the two systems are correlated in any manner if we have ρ^A is the density matrix of system A and ρ^B is the density matrix of system B, then the joint system has density matrix $\rho^A \otimes \rho^B$.
If a bipartite system has density matrix ρ^{AB} we call ρ^A the reduced density matrix of A, which corresponds to ignoring system B and whatever may happen to it. We have

$$\rho^A = Tr_B(\rho^{AB})$$

where $Tr_B(.)$ is defined to take $\tau^A \otimes \sigma^B$ into $Tr(\sigma)\tau$ and is extended to all other matrices over the tensor space by linearity.

An important subset of $\mathrm{Herm}^+_{mn}(\mathbb{C})$ is the set of *separable states*, i.e. those which can be written in the form

$$\rho = \sum_x \lambda_x \rho_1^x \otimes \rho_2^x$$

where $\lambda_x \geq 0$ and the ρ_1^x and ρ_2^x belong to $\mathrm{Herm}^+_m(\mathbb{C})$ and $\mathrm{Herm}^+_n(\mathbb{C})$ respectively. They represent those states for which there is no 'entanglement' between the subsystems. There may be correlations but these are only of a classical, probabilistic nature.

Canonicité

Let ρ_S denote the densiy matrix which corresponds to the ensemble state S. E.g. if $S = \{(p_0, |\psi_0\rangle), \ldots, (p_r, |\psi_r\rangle)\}$ then $\rho_S = \sum_i p_i |\psi_i\rangle\langle\psi_i|$.

Lemma 3.
Consider an ensemble state $S = \{(p_0, |\psi_0\rangle), \ldots, (p_r, |\psi_r\rangle)\}$, its corresponding density matrix $\rho_S = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, and a generalized measurement $\{M_k\}$. Postulate 3' on ρ_S yields the same measurement statistics as Postulate 3 on S.
Moreover suppose outcome k occurs. Consider S' the ensemble state as given from S by postulate 3, and its corresponding density matrix $\rho_{S'}$. Consider $(\rho_S)'$ the post-measurement density matrix as given from ρ_S by Postulate 3'. We have $(\rho_S)' = \rho_{S'}$.

Proof.

$$
\begin{aligned}
p(k) &= \sum_i p_i \mathrm{Tr}(M_k^\dagger M_k |\psi_i\rangle\langle\psi_i|) \\
&= \mathrm{Tr}(M_k^\dagger M_k \sum_i p_i |\psi_i\rangle\langle\psi_i|) \\
&= \mathrm{Tr}(M_k^\dagger M_k \rho_S) \\
\rho_{S'} &= \sum_i p(i|k) \frac{M_k|\psi_i\rangle\langle\psi_i|M_k^\dagger}{p(k|i)}
\end{aligned}
$$

$$\mathrm{Tr}(M_k^\dagger M_k |\psi_i\rangle\langle\psi_i|)$$

$$
\begin{aligned}
&= \sum_i p_i \frac{M_k|\psi_i\rangle\langle\psi_i|M_k^\dagger}{p(k)} \\
&= \sum_i p_i \frac{M_k|\psi_i\rangle\langle\psi_i|M_k^\dagger}{\mathrm{Tr}(M_k^\dagger M_k \rho)} \\
&= \frac{M_k \rho M_k^\dagger}{\mathrm{Tr}(M_k^\dagger M_k \rho)} \\
&= (\rho_S)' \qquad \square
\end{aligned}
$$

Lemma 4.
Consider a probability distribution over density matrices $S = \{(p_0, \rho_0), \ldots, (p_r, \rho_r)\}$, and let $\rho_S = \sum_i p_i \rho_i$, Consider a generalized measurement $\{M_k\}$. Postulate 3' on ρ_S yields the same measurement statistics as if applied on S.
Moreover suppose outcome k occurs. Consider S' the ensemble state as given from S by postulate 3', and its corresponding density matrix $\rho_{S'}$. Consider $(\rho_S)'$ the post-measurement density matrix as given from ρ_S by Postulate 3'. We have $(\rho_S)' = \rho_{S'}$.

Proof. Very similar to the one above. $\square$
Physical interpretation. The density matrix $\rho = |\psi\rangle\langle\psi|$ is an alternative, equivalent notation to the state vector $|\psi\rangle$, but this representation has the advantage of being able to hold probability distributions over states, i.e. ρ_1 with probability p_1 and ρ_2 with probability p_2 has density matrix $p_1\rho_1 + p_2\rho_2$. This representation is canonical in the sense the if $\rho \neq \sigma$, then there exists a measurement yielding different measurement statistics upon these states.

5 No-signalling

Lemma 5. *Consider ρ a bipartite state and $\mathbb{I} \otimes \widehat{\$}$ a quantum operation acting solely upon the second subsystem. We have :*

$$Tr_2(\rho) = Tr_2(\mathbb{I} \otimes \widehat{\$}(\rho))$$

Proof Say $\rho = \sum_{ij} \alpha_{ij} A_i \otimes B_j)$ and $\widehat{\$}$ has operator sum representation $\{M_k\}$.

$$\begin{aligned}
\mathrm{Tr}_2(\mathbb{I} \otimes \widehat{\$}(\rho)) &= \mathrm{Tr}_2(\mathbb{I} \otimes \widehat{\$}(\sum_{ij} \alpha_{ij} A_i \otimes B_j)) \\
&= \sum_{ij} \alpha_{ij} \mathrm{Tr}_2(\sum_k A_i \otimes M_k B_j M_k^\dagger) \\
&= \sum_{ij} \alpha_{ij} A_i \otimes \mathrm{Tr}(\sum_k M_k B_j M_k^\dagger) \\
&= \sum_{ij} \alpha_{ij} A_i \otimes \mathrm{Tr}(\sum_k M_k^\dagger M_k B_j) \\
&= \sum_{ij} \alpha_{ij} A_i \otimes \mathrm{Tr}(B_j) \\
&= \mathrm{Tr}_2(\rho)
\end{aligned}$$